# Eigenvalue, eigenvector and eigenspace

From Wikipedia, the free encyclopedia
(Redirected from Eigenvector)

In mathematics, an **eigenvector** of a transformation[1] is a non-null vector whose direction is unchanged by that transformation. The factor by which the magnitude is scaled is called the **eigenvalue** of that vector. (See Fig. 1.) Often, a transformation is completely described by its eigenvalues and eigenvectors. An **eigenspace** is a set of eigenvectors with a common eigenvalue.

These concepts play a major role in several branches of both pure and applied mathematics — appearing prominently in linear algebra, functional analysis, and to a lesser extent in nonlinear situations.

It is common to prefix any natural name for the solution with *eigen* instead of saying *eigenvector*. For example, *eigenfunction* if the eigenvector is a function, *eigenmode* if the eigenvector is a harmonic mode, *eigenstate* if the eigenvector is a quantum state, and so on (e.g. the *eigenface* example below). Similarly for the eigenvalue, e.g. *eigenfrequency* if the eigenvalue is (or determines) a frequency.

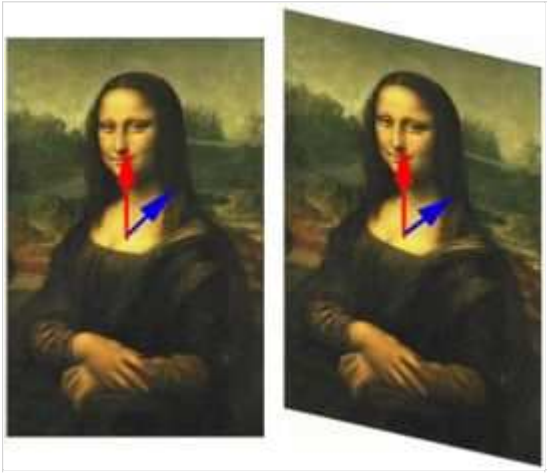

Fig. 1. In this shear transformation of the Mona Lisa, the picture was deformed in such a way that its central vertical axis (red vector) was not modified, but the diagonal vector (blue) has changed direction. Hence the red vector is an **eigenvector** of the transformation and the blue vector is not. Since the red vector was neither stretched nor compressed, its **eigenvalue** is 1. All vectors along the same vertical line are also eigenvectors, with the same eigenvalue. They form the **eigenspace** for this eigenvalue.

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## History

Nowadays, eigenvalues are usually introduced in the context of matrix theory. Historically, however, they arose in the study of quadratic forms and differential equations.

In the first half of the 18th century, Johann and Daniel Bernoulli, d'Alembert, and Euler encountered eigenvalue problems when studying the motion of a rope, which they considered to be a weightless string loaded with a number of masses. Laplace and Lagrange continued their work in the second half of the century. They realized that the eigenvalues are related to the stability of the motion. They also used eigenvalue method in their study of the solar system.[2]

Euler had also studied the rotational motion of a rigid body and discovered the importance of the principal axes. As Lagrange realized, the principal axes are the eigenvectors of the inertia matrix.[3] In the early 19th century, Cauchy saw how their work could be used to classify the quadric surfaces, and generalized it to arbitrary dimensions.[4] Cauchy also coined the term *racine caractéristique* (characteristic root) for what is now called *eigenvalue*; his term survives in *characteristic equation*.[5]

Fourier used the work of Laplace and Lagrange to solve the heat equation by separation of variables in his famous 1822 book *Théorie analytique de la chaleur*.[6] Sturm developed Fourier's ideas further and he brought them to the attention of Cauchy, who combined them with his own ideas and arrived at the fact that symmetric matrices have real eigenvalues.[4] This was extended by Hermite in 1855 to what are now called Hermitian matrices.[5] Around the same time, Brioschi proved that the eigenvalues of orthogonal matrices lie on the unit circle,[4] and Clebsch found the corresponding result for skew-symmetric matrices.[5] Finally, Weierstrass clarified an important aspect in the

stability theory started by Laplace by realizing that defective matrices can cause instability.[4]

In the meantime, Liouville had studied similar eigenvalue problems as Sturm; the discipline that grew out of their work is now called *Sturm-Liouville theory*.[7] Schwarz studied the first eigenvalue of Laplace's equation on general domains towards the end of the 19th century, while Poincaré studied Poisson's equation a few years later.[8]

At the start of the 20th century, Hilbert studied the eigenvalues of integral operators by considering them to be infinite matrices.[9] He was the first to use the German word *eigen* to denote eigenvalues and eigenvectors in 1904, though he may have been following a related usage by Helmholtz. "Eigen" can be translated as "own", "peculiar to", "characteristic" or "individual"—emphasizing how important eigenvalues are to defining the unique nature of a specific transformation. For some time, the standard term in English was "proper value", but the more distinctive term "eigenvalue" is standard today.[10]

The first numerical algorithm for computing eigenvalues and eigenvectors appeared in 1929, when Von Mises published the power method. One of the most popular methods today, the QR algorithm, was proposed independently by Francis and Kublanovskaya in 1961.[11]

# Definitions

*See also: Eigenplane*

Transformations of space—such as translation (or shifting the origin), rotation, reflection, stretching, compression, or any combination of these—may be visualized by the effect they produce on vectors. Vectors can be visualised as arrows pointing from one point to another.

- **Eigenvectors** of transformations are vectors[12] which are either left unaffected or simply multiplied by a scale factor after the transformation.
- An eigenvector's **eigenvalue** is the scale factor by which it has been multiplied.
- An **eigenspace** is a space consisting of all eigenvectors which have the same eigenvalue, along with the zero (null) vector, which itself is not an eigenvector.
- The **principal eigenvector** of a transformation is the eigenvector with the largest corresponding eigenvalue.
- The **geometric multiplicity** of an eigenvalue is the dimension of the associated eigenspace.
- The **spectrum** of a transformation on finite dimensional vector spaces is the set of all its eigenvalues.

For instance, an *eigenvector* of a rotation in three dimensions is a vector located within the axis about which the rotation is performed. The corresponding *eigenvalue* is 1 and the corresponding *eigenspace* contains all the vectors along the axis. As this is a one-dimensional space, its *geometric multiplicity* is one. This is the only eigenvalue of the *spectrum* (of this rotation) that is a real number.

# Examples

As the Earth rotates, every arrow pointing outward from the center of the Earth also rotates, except those arrows that lie on the axis of rotation. Consider the transformation of the Earth after one hour of rotation: An arrow from the center of the Earth to the Geographic South Pole would be an eigenvector of this transformation, but an arrow from the center of the Earth to anywhere on the equator would not be an eigenvector. Since the arrow pointing at the pole is not stretched by the rotation of the Earth, its eigenvalue is 1.

Another example is provided by a thin metal sheet expanding uniformly about a fixed point in such a way that the distances from any point of the sheet to the fixed point are doubled. This expansion is a transformation with eigenvalue 2. Every vector from the fixed point to a point on the sheet is an eigenvector, and the eigenspace is the set of all these vectors.

However, three-dimensional geometric space is not the only vector space. For example, consider a stressed rope fixed at both ends, like the vibrating strings of a string instrument (Fig. 2). The distances of atoms of the vibrating rope from their positions when the rope is at rest can be seen as the components of a vector in a space with as many dimensions as there are atoms in the rope.

Fig. 2. A standing wave in a rope fixed at its boundaries is an example of an eigenvector, or more precisely, an eigenfunction of the transformation giving the acceleration. As time passes, the standing wave is scaled by a sinusoidal oscillation whose frequency is determined by the eigenvalue, but its overall shape is not modified.

Assume the rope is a continuous medium. If one considers the equation for the acceleration at every point of the rope, its eigenvectors, or *eigenfunctions*, are the standing waves. The standing waves correspond to particular oscillations of the rope such that the acceleration of the rope is simply its shape scaled by a factor—this factor, the eigenvalue, turns out to be $-\omega^2$ where ω is the angular frequency of the oscillation. Each component of the vector associated with the rope is multiplied by a time-dependent factor $\sin(\omega t)$. If damping is considered, the amplitude of this oscillation decreases until the rope stops oscillating, corresponding to a complex ω. One can then associate a lifetime with the imaginary part of ω, and relate the concept of an eigenvector to the concept of resonance. Without damping, the fact that the acceleration operator (assuming a uniform density) is Hermitian leads to several important properties, such as that the standing wave patterns are orthogonal functions.

## Eigenvalue equation

Mathematically, $\mathbf{v}_\lambda$ is an eigenvector and $\lambda$ the corresponding eigenvalue of a transformation $T$ if the equation:

$$T(\mathbf{v}_\lambda) = \lambda\,\mathbf{v}_\lambda$$

is true, where $T(\mathbf{v}_\lambda)$ is the vector obtained when applying the transformation $T$ to $\mathbf{v}_\lambda$.

Suppose $T$ is a linear transformation (which means that $T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$ for all scalars *a*, *b*, and vectors **v**, **w**). Consider a basis in that vector space. Then, $T$ and $\mathbf{v}_\lambda$ can be represented relative to that basis by a matrix $A_T$—a two-dimensional array—and respectively a column vector $v_\lambda$—a one-dimensional vertical array. The eigenvalue equation in its matrix representation is written

$$A_T\,v_\lambda = \lambda\,v_\lambda$$

where the juxtaposition is matrix multiplication. Since in this circumstance the transformation $T$ and its matrix representation $A_T$ are equivalent, we can often use just $T$ for the matrix representation and the transformation. This is equivalent to a set of *n* linear equations, where *n* is the number of basis vectors in the basis set. In this equation both the eigenvalue $\lambda$ and the *n* components of $\mathbf{v}_\lambda$ are unknowns.

However, it is sometimes unnatural or even impossible to write down the eigenvalue equation in a matrix form. This occurs for instance when the vector space is infinite dimensional, for example, in the case of the rope above. Depending on the nature of the transformation $T$ and the space to which it applies, it can be advantageous to represent the eigenvalue equation as a set of differential equations. If $T$ is a differential operator, the eigenvectors are commonly called **eigenfunctions** of the differential operator representing $T$. For example, differentiation itself is a linear transformation since

$$\frac{d}{dt}(af + bg) = a\frac{df}{dt} + b\frac{dg}{dt}$$

(*f*(*t*) and *g*(*t*) are differentiable functions, and *a* and *b* are constants).

Consider differentiation with respect to *t*. Its eigenfunctions *h*(*t*) obey the eigenvalue equation:

$$\frac{dh}{dt} = \lambda h\,,$$

where *λ* is the eigenvalue associated with the function. Such a function of time is constant if λ = 0, grows proportionally to itself if λ is positive, and decays proportionally to itself if λ is negative. For example, an idealized population of rabbits breeds faster the more rabbits there are, and thus satisfies the equation with a positive lambda.

The solution to the eigenvalue equation is *g*(*t*) = exp(λ*t*), the exponential function; thus that function is an eigenfunction of the differential operator *d/dt* with the eigenvalue *λ*. If *λ* is negative, we call the evolution of *g* an exponential decay; if it is positive, an exponential growth. The value of *λ* can be any complex number. The spectrum of *d/dt* is therefore the whole complex plane. In this example the vector space in which the operator *d/dt* acts is the space of the differentiable functions of one variable. This space has an infinite dimension (because it is not possible to express every differentiable function as a linear combination of a finite number of basis functions). However, the eigenspace associated with any given eigenvalue *λ* is one dimensional. It is the set of all functions *g*(*t*) = *A*exp(λ*t*), where *A* is an arbitrary constant, the initial population at *t=0*.

# Spectral theorem

*For more details on this topic, see spectral theorem.*

According to the *spectral theorem*, the eigenvalues and eigenvectors characterize a linear transformation in a unique way. In its simplest version, the spectral theorem states that, under precise conditions, a linear transformation of a vector $\mathbf{v}$ can be expressed as the linear combination of the eigenvectors. The coefficients characterizing the linear combination are equal to the eigenvalues times the scalar product (or dot product) of the eigenvectors with the vector $\mathbf{v}$. Mathematically, it can be written as:

$$\mathcal{T}(\mathbf{v}) = \lambda_1(\mathbf{v}_1 \cdot \mathbf{v})\mathbf{v}_1 + \lambda_2(\mathbf{v}_2 \cdot \mathbf{v})\mathbf{v}_2 + \cdots$$

where $\mathbf{v}_1, \mathbf{v}_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ stand for the eigenvectors and eigenvalues of $\mathcal{T}$. The simplest case in which the theorem is valid is the case where the linear transformation is given by a real symmetric matrix or complex Hermitian matrix; more generally the theorem holds for all normal matrices.

If one defines the *n*th power of a transformation as the result of applying it *n* times in succession, one can also define polynomials of transformations. A more general version of the theorem is that any polynomial *P* of $\mathcal{T}$ is equal to:

$$P(\mathcal{T})(\mathbf{v}) = P(\lambda_1)(\mathbf{v}_1 \cdot \mathbf{v})\mathbf{v}_1 + P(\lambda_2)(\mathbf{v}_2 \cdot \mathbf{v})\mathbf{v}_2 + \cdots$$

The theorem can be extended to other functions of transformations like analytic functions, the most general case being Borel functions.

# Eigenvalues and eigenvectors of matrices

## Computing eigenvalues and eigenvectors of matrices

Suppose that we want to compute the eigenvalues of a given matrix. If the matrix is small, we can compute them symbolically using the characteristic polynomial. However, this is often impossible for larger matrices, in which case we must use a numerical method.

### Symbolic computations

*For more details on this topic, see symbolic computation of matrix eigenvalues.*

Finding eigenvalues

An important tool for describing eigenvalues of square matrices is the characteristic polynomial: saying that $\lambda$ is an eigenvalue of $A$ is equivalent to stating that the system of linear equations $(A - \lambda I)\, v = 0$ (where $I$ is the identity matrix) has a non-zero solution $v$ (an eigenvector), and so it is equivalent to the determinant:

$$\det(A - \lambda I) = 0$$

The function $p(\lambda) = \det(A - \lambda I)$ is a polynomial in $\lambda$ since determinants are defined as sums of products. This is the **characteristic polynomial** of $A$: the eigenvalues of a matrix are the zeros of its characteristic polynomial.

All the eigenvalues of a matrix $A$ can be computed by solving the equation $p_A(\lambda) = 0$. If $A$ is an $n \times n$ matrix, then $p_A$ has degree $n$ and $A$ can therefore have at most $n$ eigenvalues. Conversely, the fundamental theorem of algebra says that this equation has exactly $n$ roots (zeroes), counted with multiplicity. All real polynomials of odd degree have a real number as a root, so for odd n, every real matrix has at least one real eigenvalue. In the case of a real matrix, for even and odd $n$, the non-real eigenvalues come in conjugate pairs.

Finding eigenvectors

Once the eigenvalues λ are known, the eigenvectors can then be found by solving:

$$(A - \lambda I)v = 0$$

where v is in the null space of $A - \lambda I$

An example of a matrix with no real eigenvalues is the 90-degree clockwise rotation:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

whose characteristic polynomial is $\lambda^2 + 1$ and so its eigenvalues are the pair of complex conjugates $i$, $-i$. The associated eigenvectors are also not real.

**Numerical computations**

*For more details on this topic, see eigenvalue algorithm.*

In practice, eigenvalues of large matrices are not computed using the characteristic polynomial. Computing the polynomial becomes expensive in itself, and exact (symbolic) roots of a high-degree polynomial can be difficult to compute and express: the Abel–Ruffini theorem implies that the roots of high-degree (5 and above) polynomials cannot be expressed simply using $n$th roots. Effective numerical algorithms for approximating roots of polynomials exist, but small errors in the eigenvalues can lead to large errors in the eigenvectors. Therefore, general algorithms to find eigenvectors and eigenvalues, are iterative. The easiest method is the power method: a random vector $v$ is chosen and a sequence of unit vectors is computed as

$$\frac{Av}{\|Av\|}, \frac{A^2v}{\|A^2v\|}, \frac{A^3v}{\|A^3v\|}, \ldots$$

This sequence will almost always converge to an eigenvector corresponding to the eigenvalue of greatest magnitude. This algorithm is easy, but not very useful by itself. However, popular methods such as the QR algorithm are based on it.

## Properties

**Algebraic multiplicity**

The **algebraic multiplicity** of an eigenvalue λ of *A* is the order of λ as a zero of the characteristic polynomial of *A*; in other words, if λ is one root of the polynomial, it is the number of factors ($t - \lambda$) in the characteristic polynomial after factorization. An *n*×*n* matrix has *n* eigenvalues, counted according to their algebraic multiplicity, because its characteristic polynomial has degree *n*.

An eigenvalue of algebraic multiplicity 1 is called a "simple eigenvalue".

In an article on matrix theory, a statement like the one below might be encountered:

> "the eigenvalues of a matrix *A* are 4,4,3,3,3,2,2,1,"

meaning that the algebraic multiplicity of 4 is two, of 3 is three, of 2 is two and of 1 is one. This style is used because algebraic multiplicity is the key to many mathematical proofs in matrix theory.

Recall that above we defined the *geometric* multiplicity of an eigenvector to be the dimension of the associated eigenspace, the nullspace of λI − *A*. The algebraic multiplicity can also be thought of as a dimension: it is the dimension of the associated *generalized eigenspace* (1st sense), which is the nullspace of the matrix $(\lambda I - A)^k$ for *any sufficiently large k*. That is, it is the space of *generalized eigenvectors* (1st sense), where a generalized eigenvector is any vector which *eventually* becomes 0 if λI − *A* is applied to it enough times successively. Any eigenvector is a generalized eigenvector, and so each eigenspace is contained in the associated generalized eigenspace. This provides an easy proof that the geometric multiplicity is always less than or equal to the algebraic multiplicity. The first sense should not to be confused with generalized eigenvalue problem as stated below.

For example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

It has only one eigenvalue, namely λ = 1. The characteristic polynomial is $(\lambda - 1)^2$, so this eigenvalue has algebraic multiplicity 2. However, the associated eigenspace is the axis usually called the *x* axis, spanned by the unit vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so the geometric multiplicity is only 1.

Generalized eigenvectors can be used to calculate the Jordan normal form of a matrix (see discussion below). The fact that Jordan blocks in general are not diagonal but nilpotent is directly related to the distinction between eigenvectors and generalized eigenvectors.

**Decomposition theorems for general matrices**

The **decomposition theorem** is a version of the spectral theorem in the particular case of matrices. This theorem is usually introduced in terms of coordinate transformation. If *U* is an invertible matrix, it can be seen as a transformation from one coordinate system to another, with the columns of *U* being the components of the new basis vectors within the old basis set. In this new system the coordinates of the vector *v* are labeled *v*'. The latter are obtained from the coordinates *v* in the original coordinate system by the relation *v*' = *Uv* and, the other way around, we have $v = U^{-1}v'$. Applying successively *v*' = *Uv*, *w*' = *Uw* and $U^{-1}U = I$, to the relation *Av* = *w* defining the matrix multiplication provides *A*'*v*' = *w*' with $A' = UAU^{-1}$, the representation of *A* in the new basis. In this situation, the matrices *A* and *A*' are said to be similar.

The decomposition theorem states that, if one chooses as columns of $U^{-1}$ *n* linearly independent eigenvectors of

$A$, the new matrix $A' = UAU^{-1}$ is diagonal and its diagonal elements are the eigenvalues of $A$. If this is possible the matrix $A$ is *diagonalizable*. An example of non-diagonalizable matrix is given by the matrix $A$ above. There are several generalizations of this decomposition which can cope with the non-diagonalizable case, suited for different purposes:

- the Schur triangular form states that any matrix is unitarily equivalent to an upper triangular one;
- the singular value decomposition, $A = U\Sigma V^*$ where Σ is diagonal with $U$ and $V$ unitary matrices. The diagonal entries of $A = U\Sigma V^*$ are nonnegative; they are called the singular values of $A$. This can be done for non-square matrices as well;
- the Jordan normal form, where $A = X\Lambda X^{-1}$ where Λ is not diagonal but block-diagonal. The number and the sizes of the Jordan blocks are dictated by the geometric and algebraic multiplicities of the eigenvalues. The Jordan decomposition is a fundamental result. One might glean from it immediately that a square matrix is described completely by its eigenvalues, including multiplicity, up to similarity. This shows mathematically the important role played by eigenvalues in the study of matrices;
- as an immediate consequence of Jordan decomposition, any matrix $A$ can be written *uniquely* as $A = S + N$ where $S$ is diagonalizable, $N$ is nilpotent (i.e., such that $N^q=0$ for some $q$), and $S$ commutes with $N$ ($SN=NS$).

**Some other properties of eigenvalues**

The spectrum is invariant under similarity transformations: the matrices $A$ and $P^{-1}AP$ have the same eigenvalues for any matrix $A$ and any invertible matrix $P$. The spectrum is also invariant under transposition: the matrices $A$ and $A^T$ have the same eigenvalues.

Since a linear transformation on finite dimensional spaces is bijective iff it is injective, a matrix is invertible if and only if zero is not an eigenvalue of the matrix.

Some more consequences of the Jordan decomposition are as follows:

- a matrix is diagonalizable if and only if the algebraic and geometric multiplicities coincide for all its eigenvalues. In particular, an $n \times n$ matrix which has $n$ different eigenvalues is always diagonalizable;
- the vector space on which the matrix acts can be viewed as a direct sum of its invariant subspaces span by its generalized eigenvectors. Each block on the diagonal corresponds to a subspace in the direct sum. When a block is diagonal, its invariant subspace is an eigenspace. Otherwise it is a generalized eigenspace, defined above;
- Since the trace, or the sum of the elements on the main diagonal of a matrix, is preserved by unitary equivalence, the Jordan normal form tells us that it is equal to the sum of the eigenvalues;
- Similarly, because the eigenvalues of a triangular matrix are the entries on the main diagonal, the determinant equals the product of the eigenvalues (counted according to algebraic multiplicity).

The location of the spectrum for a few subclasses of normal matrices are:

- All eigenvalues of a Hermitian matrix ($A = A^*$) are real. Furthermore, all eigenvalues of a positive-definite matrix ($v^*Av > 0$ for all vectors $v$) are positive;
- All eigenvalues of a skew-Hermitian matrix ($A = -A^*$) are purely imaginary;
- All eigenvalues of a unitary matrix ($A^{-1} = A^*$) have absolute value one;

Suppose that $A$ is an $m \times n$ matrix, with $m \le n$, and that $B$ is an $n \times m$ matrix. Then $BA$ has the same eigenvalues as $AB$ plus $n - m$ eigenvalues equal to zero.

Each matrix can be assigned an operator norm, which depends on the norm of its domain. The operator norm of a

square matrix is an upper bound for the moduli of its eigenvalues, and thus also for its spectral radius. This norm is directly related to the power method for calculating the eigenvalue of largest modulus given above. For normal matrices, the operator norm induced by the Euclidean norm is the largest moduli among its eigenvalues.

### Conjugate eigenvector

A **conjugate eigenvector** or **coneigenvector** is a vector sent after transformation to a scalar multiple of its conjugate, where the scalar is called the **conjugate eigenvalue** or **coneigenvalue** of the linear transformation. The coneigenvectors and coneigenvalues represent essentially the same information and meaning as the regular eigenvectors and eigenvalues, but arise when an alternative coordinate system is used. The corresponding equation is

$$Av = \lambda v^*.$$

For example, in coherent electromagnetic scattering theory, the linear transformation *A* represents the action performed by the scattering object, and the eigenvectors represent polarization states of the electromagnetic wave. In optics, the coordinate system is defined from the wave's viewpoint, known as the Forward Scattering Alignment (FSA), and gives rise to a regular eigenvalue equation, whereas in radar, the coordinate system is defined from the radar's viewpoint, known as the Back Scattering Alignment (BSA), and gives rise to a coneigenvalue equation.

### Generalized eigenvalue problem

A **generalized eigenvalue problem** (2nd sense) is of the form

$$Av = \lambda Bv$$

where *A* and *B* are matrices. The **generalized eigenvalues** (2nd sense) λ can be obtained by solving the equation

$$\det(A - \lambda B) = 0.$$

The set of matrices of the form $A - \lambda B$, where λ is a complex number, is called a *pencil*. If *B* is invertible, then the original problem can be written in the form

$$B^{-1}Av = \lambda v$$

which is a standard eigenvalue problem. However, in most situations it is preferable not to perform the inversion, and solve the generalized eigenvalue problem as stated originally.

An example is provided by the molecular orbital application below.

### Entries from a ring

In the case of a square matrix *A* with entries in a ring, λ is called a **right eigenvalue** if there exists a nonzero column vector *x* such that $Ax=\lambda x$, or a **left eigenvalue** if there exists a nonzero row vector *y* such that $yA=y\lambda$.

If the ring is commutative, the left eigenvalues are equal to the right eigenvalues and are just called eigenvalues. If not, for instance if the ring is the set of quaternions, they may be different.

## Infinite-dimensional spaces

If the vector space is infinite dimensional, the notion of eigenvalues can be generalized to the concept of spectrum. The spectrum is the set of scalars λ for which, $(T-\lambda)^{-1}$, is not defined; that is, such that $T-\lambda$ has no bounded inverse.

Clearly if $\lambda$ is an eigenvalue of $T$, $\lambda$ is in the spectrum of $T$. In general, the converse is not true. There are operators on Hilbert or Banach spaces which have no eigenvectors at all. This can be seen in the following example. The bilateral shift on the Hilbert space $\ell^2(\mathbf{Z})$ (the space of all sequences of scalars $\ldots a_{-1}, a_0, a_1, a_2, \ldots$ such that $\cdots + |a_{-1}|^2 + |a_0|^2 + |a_1|^2 + |a_2|^2 + \cdots$ converges) has no eigenvalue but has spectral values.

In infinite-dimensional spaces, the spectrum of a bounded operator is always nonempty. This is also true for an unbounded self adjoint operator. Via its spectral measures, the spectrum of any self adjoint operator, bounded or otherwise, can be decomposed into absolutely continuous, pure point, and singular parts. (See Decomposition of spectrum.)

The exponential growth or decay provides an example of a continuous spectrum, as does the vibrating string example illustrated above. The hydrogen atom is an example where both types of spectra appear. The bound states of the hydrogen atom correspond to the discrete part of the spectrum while the ionization processes are described by the continuous part. Fig. 3 exemplifies this concept in the case of the Chlorine atom.

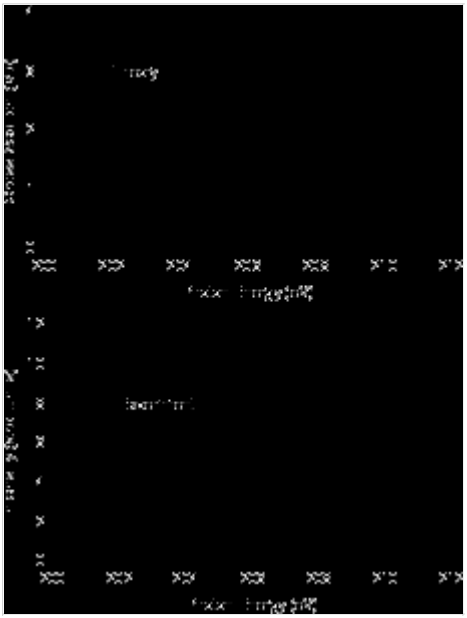

Fig. 3.Absorption spectrum (cross section) of atomic Chlorine. The sharp lines obtained in theory correspond to the discrete spectrum (Rydberg series) of the Hamiltonian; the broad structure on the right is associated with the continuous spectrum (ionization). The corresponding experimental results have been obtained by measuring the intensity of X-rays absorbed by a gas of atoms as a function of the incident photon energy in eV.[13]

## Applications

Schrödinger equation

An example of an eigenvalue equation where the transformation $T$ is represented in terms of a differential operator is the time-independent Schrödinger equation in quantum mechanics:

$$H\Psi_E = E\Psi_E$$

where *H*, the Hamiltonian, is a second-order differential operator and $\Psi_E$, the wavefunction, is one of its eigenfunctions corresponding to the eigenvalue *E*, interpreted as its energy.

However, in the case where one is interested only in the bound state solutions of the Schrödinger equation, one looks for $\Psi_E$ within the space of square integrable functions. Since this space is a Hilbert space with a well-defined scalar product, one can introduce a basis set in which $\Psi_E$ and *H* can be represented as a one-dimensional array and a matrix respectively. This allows one to represent the Schrödinger equation in a matrix form. (Fig. 4 presents the lowest eigenfunctions of the Hydrogen atom Hamiltonian.)

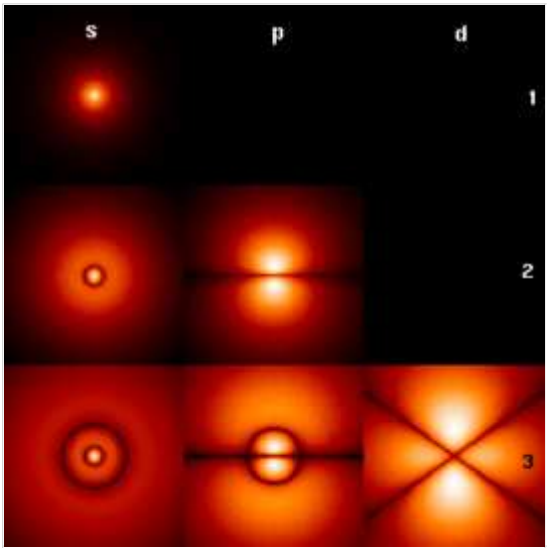


Fig. 4. The wavefunctions associated with the bound states of an electron in a hydrogen atom can be seen as the eigenvectors of the hydrogen atom Hamiltonian as well as of the angular momentum operator. They are associated with eigenvalues interpreted as their energies (increasing downward: *n*=1,2,3,...) and angular momentum (increasing across: *s*, *p*, *d*,...). The illustration shows the square of the absolute value of the wavefunctions. Brighter areas correspond to higher probability density for a position measurement. The center of each figure is the atomic nucleus, a proton.

The Dirac notation is often used in this context. A vector, which represents a state of the system, in the Hilbert space of square integrable functions is represented by $|\Psi_E\rangle$. In this notation, the Schrödinger equation is:

$$H|\Psi_E\rangle = E|\Psi_E\rangle$$

where $|\Psi_E\rangle$ is an **eigenstate** of *H*. It is a self adjoint operator, the infinite dimensional analog of Hermitian matrices (*see Observable*). As in the matrix case, in the equation above $H|\Psi_E\rangle$ is understood to be the vector obtained by application of the transformation *H* to $|\Psi_E\rangle$.

Molecular orbitals

In quantum mechanics, and in particular in atomic and molecular physics, within the Hartree-Fock theory, the atomic and molecular orbitals can be defined by the eigenvectors of the Fock operator. The corresponding eigenvalues are interpreted as ionization potentials via Koopmans' theorem. In this case, the term eigenvector is used in a somewhat more general meaning, since the Fock operator is explicitly dependent on the orbitals and their eigenvalues. If one wants to underline this aspect one speaks of *implicit eigenvalue equation*. Such equations are usually solved by an iteration procedure, called in this case self-consistent field method. In quantum chemistry, one often represents the Hartree-Fock equation in a non-orthogonal basis set. This particular representation is a generalized eigenvalue problem called Roothaan equations.

Factor analysis

In factor analysis, the eigenvectors of a covariance matrix correspond to factors, and eigenvalues to factor loadings. Factor analysis is a statistical technique used in the social sciences and in marketing, product management, operations research, and other applied sciences that deal with large quantities of data. The objective is to explain most of the variability among a number of observable random variables in terms of a smaller number of unobservable random variables called factors. The observable random variables are modeled as linear combinations of the factors, plus "error" terms.

Eigenfaces

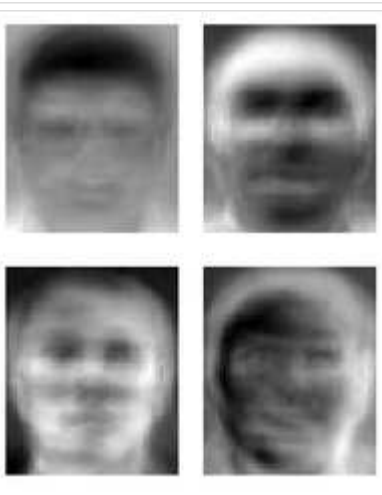

Fig. 5. Eigenfaces as examples of eigenvectors

In image processing, processed images of faces can be seen as vectors whose components are the brightnesses of each pixel. The dimension of this vector space is the number of pixels. The eigenvectors of the covariance matrix associated to a large set of normalized pictures of faces are called eigenfaces. They are very useful for expressing any face image as a linear combination of some of them. Eigenfaces provide a means of applying data compression to faces for identification purposes.

Tensor of inertia

In mechanics, the eigenvectors of the inertia tensor define the principal axes of a rigid body. The tensor of inertia is a key quantity required in order to determine the rotation of a rigid body around its center of mass.

Stress tensor

In solid mechanics, the stress tensor is symmetric and so can be decomposed into a diagonal tensor with the eigenvalues on the diagonal and eigenvectors as a basis. Because it is diagonal, in this orientation, the stress tensor has no shear components; the components it does have are the principal components.

Eigenvalues of a graph

In spectral graph theory, an eigenvalue of a graph is defined as an eigenvalue of the graph's adjacency matrix $A$, or (increasingly) of the graph's Laplacian matrix $I - T^{-1/2}AT^{-1/2}$, where $T$ is a diagonal matrix holding the degree of each vertex, and in $T^{-1/2}$, 0 is substituted for $0^{-1/2}$. The principal eigenvector of a graph is used to measure the centrality of its vertices. An example is Google's PageRank algorithm. The principal eigenvector of a modified adjacency matrix of the www-graph gives the page ranks as its components. The two eigenvectors with largest positive eigenvalues can also be used as *x*- and *y*-coordinates of vertices in drawing the graph via spectral layout methods.

# Notes

1. **^** In this context, only linear transformations from a vector space to itself are considered.
2. **^** See Hawkins (1975), §2; Kline (1972), pp. 807+808.
3. **^** See Hawkins (1975), §2.
4. ^ ***a b c d*** See Hawkins (1975), §3.
5. ^ ***a b c*** See Kline (1972), pp. 807+808.
6. **^** See Kline (1972), p. 673.
7. **^** See Kline (1972), pp. 715+716.
8. **^** See Kline (1972), pp. 706+707.
9. **^** See Kline (1972), p. 1063.
10. **^** See Aldrich (2006).
11. **^** See Golub and Van Loan (1996), §7.3; Meyer (2000), §7.3.
12. **^** Since all linear transformations leave the zero vector unchanged, it is not considered an eigenvector.
13. **^** T. W Gorczyca, Auger Decay of the Photoexcited Inner Shell Rydberg Series in Neon, Chlorine, and Argon, Abstracts of the 18th International Conference on X-ray and Inner-Shell Processes, Chicago, August 23-27 (1999).

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## External links

Wikibooks has a book on the topic of *Algebra/Eigenvalues and eigenvectors*

- Videos of MIT Linear Algebra Course, spring 2005 (http://ocw.mit.edu/OcwWeb/Mathematics/18-06Spring-2005/VideoLectures/index.htm) - See Lecture *Eigenvalues and Eigenvectors*
- MathWorld: Eigenvector (http://mathworld.wolfram.com/Eigenvector.html)
- ARPACK (http://www.caam.rice.edu/software/ARPACK/) is a collection of FORTRAN subroutines for solving large scale eigenvalue problems
- Eigenvalue (of a matrix) (http://planetmath.org/?op=getobj&from=objects&id=4397) on PlanetMath
- Online calculator for Eigenvalues and Eigenvectors (http://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert.htm)
- Online Matrix Calculator (http://www.bluebit.gr/matrix-calculator/) Calculates eigenvalues, eigenvectors and other decompositions of matrices online
- Vanderplaats Research and Development (http://www.vrand.com) - Provides the SMS (http://www.vrand.com) eigenvalue solver for Structural Finite Element. The solver is in the *GENESIS* (http://www.vrand.com/Genesis.html) program as well as other commercial programs. SMS can be easily use with MSC.Nastran or NX/Nastran via DMAPs.
- What are Eigen Values? (http://www.physlink.com/education/AskExperts/ae520.cfm) from PhysLink.com's "Ask the Experts"

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